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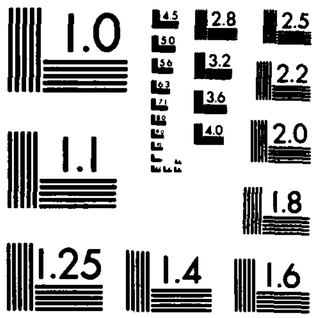

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THE CAUCHY PROBLEM IN ONE-DIMENSIONAL NONLINEAR VISCOELASTICITY

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ABSTRACT

The authors ~~We~~ study the initial value problem for a nonlinear hyperbolic Volterra equation which models the motion of an unbounded viscoelastic bar. Under physically motivated assumptions, we establish the existence of a unique, globally defined, classical solution provided the initial data are sufficiently smooth and small. They ~~We~~ also discuss boundedness and asymptotic behavior. Their ~~Our~~ analysis is based on energy estimates in conjunction with properties of strongly positive definite kernels.

AMS (MOS) Subject Classifications: 35L70, 45K05, 73F15

Key Words: nonlinear viscoelasticity, hyperbolic equation, initial value problem, classical solution, global existence, decay, energy estimates, Volterra operator, strongly positive definite kernel, resolvent kernel, Laplace transform.

Work Unit Number 1 - Applied Analysis

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SIGNIFICANCE AND EXPLANATION

We study the initial value problem for a nonlinear integrodifferential equation of Volterra type which models the motion of an unbounded viscoelastic bar. If the kernel in the integral term vanishes identically (which corresponds to the case of an elastic bar), the equation reduces to an undamped quasilinear wave equation which does not generally have globally defined smooth solutions - no matter how smooth and small the initial data are - due to the formation of shock waves. Under physically natural assumptions on the kernel (which exclude the trivial case), the integral term has a damping effect and prevents the development of shocks if the initial data and forcing function are suitably small.

We here establish the existence of a unique, globally defined, classical solution provided the given data are sufficiently smooth and small. Moreover, we show that first and second order partial derivatives of the solution decay to zero uniformly as time tends to infinity. Analogous results are already known for bounded bars with appropriate boundary conditions, but the proofs make crucial use of certain Poincaré inequalities which are not valid for an unbounded bar. As far as local existence of solutions is concerned, it is easy to circumvent this difficulty. However, some substantial modifications are needed to show that local solutions can be continued globally. Our analysis is based on energy estimates in conjunction with properties of Volterra integral kernels.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

THE CAUCHY PROBLEM IN ONE-DIMENSIONAL NONLINEAR VISCOELASTICITY

W. J. Hrusa[1,2] and J. A. Nohel[1]

1. Introduction

The aim of this paper is to establish global existence and decay of classical solutions to the Cauchy problem

$$u_{tt}(x,t) = \phi(u_x(x,t))_x + \int_0^t a'(t-\tau)\psi(u_x(x,\tau))_x d\tau + f(x,t), \quad -\infty < x < \infty, \ t > 0 , \tag{1.1}$$

$$u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \quad -\infty < x < \infty \ , \tag{1.2}$$

for suitably smooth and small data u_0, u_1, f. Here ϕ, ψ, and a are assigned smooth functions, subscripts x and t indicate partial differentiation, and a prime denotes the derivative of a function of a single variable. Throughout this paper all derivatives should be interpreted in the distributional sense. Moreover, when we speak of a solution we always mean a classical solution.

The above problem serves as a model for the motion of an unbounded, homogeneous, viscoelastic bar. On physical grounds, it is natural to assume that a is positive, decreasing, and convex, with* $a(t) \to 0$ as $t \to \infty$, and that

$$\phi(0) = \psi(0) = 0, \ \phi'(0) > 0, \ \psi'(0) > 0, \ \phi'(0) - a(0)\psi'(0) > 0 \ . \tag{1.3}$$

We refer to our survey paper [5] for a discussion of the physical interpretation of (1.1) and a much more complete summary of previous related work. (In addition, [5] contains a proof of Theorem 1.1 in the (much simpler) special case of an exponential kernel.)

Observe that if $a' \equiv 0$, then (1.1) reduces to the quasilinear wave equation

$$u_{tt} = \phi(u_x)_x + f \ . \tag{1.4}$$

It is well known that (1.4), (1.2) does not generally have a global (in time) smooth

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* Since a' (rather than a) appears in the equation of motion, a constant can be added to the kernel a without affecting (1.1). The normalization $a(\infty) = 0$ is convenient for our purposes. The reader is cautioned that other normalizations are also used.

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solution, no matter how smooth and small the data are. (See, for example, [8] and [11].) As explained in [5], the memory term in (1.1) has a damping effect if $a' \nequiv 0$ and the appropriate sign conditions are satisfied. However, this dampimg mechanism is quite subtle and globally defined smooth solutions should be expected only if the data are suitably small.

Our main interest here is in global phenomena. If the kernel is sufficiently regular, it is a more or less routine matter to establish local existence of solutions to (1.1), (1.2). However, the question of securing suitable global estimates is considerably more delicate, especially for an unbounded bar.

Dafermos and Nohel [2] have established small-data global existence theorems for analogous initial-boundary value problems corresponding to the motions of bounded viscoelastic bodies. (They treat Neumann, Dirichlet, and mixed conditions.) However, their argument makes crucial use of various Poincaré inequalities and consequently is not applicable to (1.1), (1.2).

Equations of the form (1.1) with $\psi \equiv \phi$ have been studied by MacCamy [10], Dafermos and Nohel [1], Staffans [16], and Hattori [4]. Small-data global existence theorems (for bounded and unbounded bodies) are given in [10], [1], and [16]. Nonexistence of global solutions for certain large data (of arbitrary smoothness) is established in [4]. (See also [3], [12], and [14] for some related nonexistence results.) If $\psi \equiv \phi$, equation (1.1) admits certain estimates which do not carry over to the general case with ψ different from ϕ. However, we know of no physical motivation for the restriction $\psi \equiv \phi$.

We here establish global existence and decay of classical solutions to (1.1), (1.2) (with ψ different from ϕ), under assumptions quite similar to those used in [2] for the case of a bounded bar. The proof combines certain estimates of Dafermos and Nohel [2] (which remain valid for unbounded bars) with a variant of a procedure introduced by MacCamy in [9], [10].

As in [2], we assume

$$a, a', a'' \in L^1(0,\infty), \quad a \text{ is strongly positive definite .} \tag{1.5}$$

(We note that twice continuously differentiable a with $(-1)^j a^{(j)}(t) \geq 0$ for all $t > 0$,

$j = 0,1,2$; $a' \not\equiv 0$ are automatically strongly positive definite. See [13].) In addition, we require

$$\int_0^\infty t|a(t)|dt < \infty, \quad \widehat{a'}(z) \neq 0 \quad \forall\, z \in \Pi \quad , \tag{1.6}$$

where $\hat{}$ denotes the Laplace transform and* $\Pi := \{z \in C : \mathrm{Re}\, z > 0\}$. The additional conditions (1.6) are not terribly restrictive. (This will be discussed further at the end of Section 2.)

Regarding u_0, u_1, and f, we assume

$$u_0 \in L^2_{\ell oc}(\mathbb{R}),\ u_0',\ u_0'',\ u_0''',\ u_1,\ u_1',\ u_1'' \in L^2(\mathbb{R}) \quad , \tag{1.7}$$

$$f,\ f_x,\ f_t \in C([0,\infty);\ L^2(\mathbb{R})) \cap L^\infty([0,\infty);\ L^2(\mathbb{R})) \quad , \tag{1.8}$$

$$f \in L^1([0,\infty);\ L^2(\mathbb{R})),\ f_x,\ f_t,\ f_{xt} \in L^2([0,\infty);\ L^2(\mathbb{R})) \quad . \tag{1.9}$$

In order to keep things reasonably simple, we have made our hypotheses on f slightly stronger than necessary. Several similar conditions can be used in place of (1.9). (In fact, we could assume that (1.8) holds and that f is a sum of several functions each of which satisfies a condition in the spirit of (1.9).) Finally, to measure the size of the data we define

$$U_0(u_0,u_1) := \int_{-\infty}^\infty \{u_0'(x)^2 + u_0''(x)^2 + u_0'''(x)^2 + u_1(x)^2 + u_1'(x)^2 + u_1''(x)^2\}dx \tag{1.10}$$

and

$$\begin{aligned} F(f) := \sup_{t\geq 0} \int_{-\infty}^\infty \{f^2 + f_x^2 + f_t^2\}(x,t)dx \\ + \int_0^\infty \int_{-\infty}^\infty \{f_x^2 + f_t^2 + f_{xt}^2\}(x,t)dxdt + \left(\int_0^\infty \left(\int_{-\infty}^\infty f(x,t)^2 dx\right)^{1/2} dt\right)^2 \quad . \end{aligned} \tag{1.11}$$

Our main result is

<u>Theorem 1.1</u>: Assume that ϕ, $\psi \in C^3(\mathbb{R})$ and that (1.3), (1.5), and (1.6) hold. Then, there exists a constant $\mu > 0$ such that for every u_0, u_1, and f satisfying (1.7), (1.8), (1.9), and

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* The symbol $:=$ indicates an equality in which the left hand side is defined by the right hand side.

$$U_0(u_0,u_1) + F(f) < \mu^2 \quad , \tag{1.12}$$

the initial value problem (1.1), (1.2) has a unique solution $u \in C^2(\mathbb{R} \times [0,\infty))$ with

$$\begin{gathered} u_x,\ u_t,\ u_{xx},\ u_{xt},\ u_{tt},\ u_{xxx},\ u_{xxt},\ u_{xtt}, \\ u_{ttt} \in C([0,\infty);\ L^2(\mathbb{R})) \cap L^\infty([0,\infty);\ L^2(\mathbb{R})) \quad . \end{gathered} \tag{1.13}$$

In addition,

$$u_{xx}, u_{xt}, u_{tt}, u_{xxx}, u_{xxt}, u_{xtt}, u_{ttt} \in L^2([0,\infty);\ L^2(\mathbb{R})) \quad , \tag{1.14}$$

$$u_{xx},\ u_{xt},\ u_{tt} \to 0 \quad \text{in} \quad L^2(\mathbb{R}) \quad \text{as} \quad t \to \infty \quad , \tag{1.15}$$

and

$$u_x, u_t, u_{xx}, u_{xt}, u_{tt} \to 0 \quad \text{uniformly on} \quad \mathbb{R} \quad \text{as} \quad t \to \infty \quad . \tag{1.16}$$

Remark 1.1: Assumption (1.5) implies that $a' \in AC[0,\infty)$. There are indications that for certain viscoelastic materials $a'(t) \sim - t^{\alpha-1}$ as $t \to 0$, with $0 < \alpha < 1$. Recently, Hrusa and Renardy [6] have studied equations of the form (1.1) under assumptions on a which permit such singularities in a'. For the case of a bounded bar, they establish local as well as global existence theorems. For (1.1), (1.2), they have local results, but no global results. (Again, this is due to the lack of Poincaré inequalities on all of space.) Unfortunately, the techniques which we employ here to estimate lower order derivatives make essential use of the assumption $a'' \in L^1(0,\infty)$, and consequently we cannot handle the case when a' is singular.

Remark 1.2: Dafermos and Nohel [2] mention possible extensions of their results to problems involving motions of multidimensional viscoelastic bodies. The same comments apply here. In particular, if the kernel is a scalar multiple of the identity, then a straightforward (but tedious) modification of the proof of Theorem 1.1 can be used to establish global existence of solutions for small data. For a three dimensional problem, estimates on derivates of u through order 4 (rather than through order 3 as in one dimension) would be required. However, the case of a general matrix-valued kernel A is considerably more complicated. It is not very hard to state implicit assumptions on A under which global existence could be established. The difficulty lies in determining simple and direct conditions on A which would guarantee that these assumptions are satisfied.

The remaining two sections of this paper are devoted to the proof of Theorem 1.1. Section 2 contains some preliminary material on local solutions as well as properties of the kernel a and several related resolvent kernels. The actual proof is presented in Section 3.

Acknowledgement: We are indebted to Professor R. L. Wheeler for some valuable suggestions which enabled us to weaken our original assumptions on a.

2. Preliminaries

We begin by stating a local existence result for (1.1), (1.2).

Lemma 2.1: Assume that $\phi, \psi \in C^3(\mathbb{R})$, $a, a', a'' \in L^1_{\ell oc}[0,\infty)$, and that there exists a constant $\underline{\phi} > 0$ such that

$$\phi'(\xi) \geq \underline{\phi} \quad \forall\, \xi \in \mathbb{R} . \tag{2.1}$$

Let u_0, u_1, and f satisfying (1.7), (1.8), and $f_{xt} \in L^1_{\ell oc}([0,\infty); L^2(\mathbb{R}))$ be given. Then, the initial value problem (1.1), (1.2) has a unique local solution u, defined on a maximal time interval $[0,T_0)$, with

$$u_x, u_t, u_{xx}, u_{xt}, u_{tt}, u_{xxx}, u_{xxt}, u_{xtt}, u_{ttt} \in C([0,T_0); L^2(\mathbb{R})) . \tag{2.2}$$

Moreover, if

$$\sup_{t\in[0,T_0)} \int_{-\infty}^{\infty} \{u_x^2 + u_t^2 + u_{xx}^2 + u_{xt}^2 + u_{tt}^2 + u_{xxx}^2 + u_{xxt}^2 + u_{xtt}^2 + u_{ttt}^2\}(x,t)\,dx < \infty , \tag{2.3}$$

then $T_0 = \infty$.

Remark 2.1: The Sobolev embedding theorem and (2.2) imply $u \in C^2(\mathbb{R} \times [0,T_0))$.

The proof of Lemma 2.1 is almost identical to the proof of Theorem 2.1 of [2]. Therefore, we omit the details. The only significant difference is that certain additional estimates are needed for lower order derivatives. As far as local existence is concerned, this causes no difficulties; one simply expresses the lower order derivatives in terms of initial conditions and time integrals of higher order derivatives. (Of course, such a procedure yields time-dependent bounds and cannot be used to obtain global estimates.)

The integrability properties of several resolvent kernels associated with a are crucial to our analysis of (1.1), (1.2). Therefore, we briefly recall a few basic concepts. Let $b \in L^1_{\ell oc}[0,\infty)$ be given and consider the linear (scalar) Volterra equation

$$y(t) + \int_0^t b(t-\tau)y(\tau)d\tau = g(t), \quad t \geq 0 \quad . \tag{2.4}$$

For each $g \in L^1_{\ell oc}[0,\infty)$, equation (2.4) has a unique solution $y \in L^1_{\ell oc}[0,\infty)$. Moreover, this solution is given by

$$y(t) = g(t) + \int_0^t p(t-\tau)g(\tau)d\tau, \quad t \geq 0 \quad , \tag{2.5}$$

where p is the unique solution of the resolvent equation

$$p(t) + \int_0^t b(t-\tau)p(\tau)d\tau = -b(t), \quad t \geq 0 \quad . \tag{2.6}$$

A classical theorem of Paley and Wiener states that if b belongs to $L^1(0,\infty)$, then the resolvent kernel p belongs to $L^1(0,\infty)$ if and only if $1 + \hat{b}(z)$ does not vanish for any $z \in \Pi$.

We shall also make use of several basic properties of strongly positive definite kernels. A function $a \in L^1_{\ell oc}[0,\infty)$ is said to be positive definite if

$$\int_0^t y(s) \int_0^s a(s-\tau)y(\tau)d\tau ds \geq 0 \quad \forall\, t \geq 0 \quad , \tag{2.7}$$

for every $y \in C[0,\infty)$; a is called strongly positive definite if there exists a constant $c > 0$ such that the function defined by $a(t) - ce^{-t}$, $t \geq 0$, is positive definite. As the terminology suggests, strongly positive definite implies positive definite.

These definitions are generally not very easy to check directly. For our purposes here, it is useful to know that if a belongs to $L^1(0,\infty)$ then a is strongly positive definite if and only if there exists a constant $C > 0$ such that

$$\mathrm{Re}\,\hat{a}(i\omega) \geq \frac{C}{\omega^2+1} \quad \forall\, \omega \in \mathbb{R} \ . \tag{2.8}$$

Moreover, if a positive definite function is sufficiently regular then statements can be made concerning its pointwise behavior near zero. In particular, (1.5) implies

$$a(0) > 0 \quad , \quad a'(0) < 0 \ . \tag{2.9}$$

(That (1.5) implies $a(0) > 0$ follows easily from $a(0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{Re}\,\hat{a}(i\omega)d\omega$ and (2.8). To see that (1.5) implies $a'(0) < 0$, observe that $\lim_{\omega\to\infty} \omega^2\, \mathrm{Re}\,\hat{a}(i\omega) = -a'(0)$, as can be verified using two integrations by parts and the Riemann-Lebesgue lemma. This limit must be strictly positive by (2.8).) See, for example, [13] and [15] for more information on these matters.

The kernel k defined by

$$\phi'(0)k(t) + \int_0^t a'(t-\tau)\psi'(0)k(\tau)d\tau = -\psi'(0)a'(t), \quad t \geq 0 \ , \tag{2.10}$$

can be used to express u_{xx} (or u_{xxx}) in terms of u_{tt} (or u_{xtt}) and small correction terms through an equation quite similar to (2.5). (The fact that the leading coefficient in (2.10) is different from one does not affect the representation formulas (2.4), (2.5), (2.6) in a significant way. We can simply divide through by $\phi'(0)$ since $\phi'(0) > 0$.) Thus, if $k \in L^1(0,\infty)$, bounds on u_{xx} (or u_{xxx}) can be inferred from bounds on u_{tt} (or u_{xtt}). Using the Paley-Wiener theorem, (1.3), (1.5), and properties of strongly positive definite kernels, one can establish

<u>Lemma 2.2</u>: Assume that (1.3) and (1.5) hold. Then, the solution k of (2.10) belongs to $L^1(0,\infty)$.

This lemma was used previously by Dafermos and Nohel. See Lemma 3.2 of [2] for the proof.

In order to simplify the formulas in Lemmas 2.3 and 2.4 below, we assume, without loss of generality, that

$$a'(0) = -1 \ . \tag{2.11}$$

(Since $a'(0) < 0$, we can multiply ψ by $-a'(0)$ and divide a by $-a'(0)$ to achieve

(2.11). Such a change does not affect the assumptions of Theorem 1.1.)

Let r denote the resolvent kernel associated with $-a''$, i.e. the solution of

$$r(t) - \int_0^t a''(t-\tau)r(\tau)d\tau = a''(t), \quad t > 0 \quad . \tag{2.12}$$

It is not hard to see that $r \notin L^1(0,\infty)$, since $a'(0) = -1$. However, it follows from (1.5) and (1.6) that r is the sum of a positive constant and an L^1-function. More precisely, we have

__Lemma 2.3:__ Assume that (1.5), (1.6), and (2.11) hold. Then, the solution r of (2.12) satisfies

$$r(t) = \frac{1}{a(0)} + R(t) \quad \forall\, t > 0 \quad , \tag{2.13}$$

where $R \in L^1(0,\infty)$.

__Proof:__ Formally taking Laplace transforms in (2.12) and using (2.13), we find, after a simple computation that

$$\hat{R}(z) = \frac{\hat{a}(z)a(0)^{-1} + \widehat{a'}(z)}{-\widehat{a'}(z)} \quad , \quad z \in \Pi \quad . \tag{2.14}$$

Since $\widehat{a'}$ does not vanish on Π, it is clear that $\hat{R}$ is locally analytic on Π in the sense of Definition 2.1 of [7] (with $\rho(t) \equiv 1$). Observe that for z near infinity, we have

$$\hat{R}(z) = \frac{\widehat{a''}(z)}{1 - \widehat{a''}(z)} - \frac{1}{a(0)z} \quad . \tag{2.15}$$

Thus $\hat{R}$ is locally analytic at infinity and $\hat{R}(\infty) = 0$. Therefore, Proposition 2.3 of [7] implies that $\hat{R}$ is the Laplace transform of a function $R \in L^1(0,\infty)$, and the lemma follows easily.

■

It is convenient to define another kernel M by

$$M(t) := -\int_t^\infty R(s)ds \quad \forall\, t > 0 \quad . \tag{2.16}$$

For our proof of Theorem 1.1, it is essential to know that $M \in L^1(0,\infty)$ and $M(0) < 1$.

__Lemma 2.4:__ Assume that (1.5), (1.6), and (2.11) hold. Then, the kernel M defined by

(2.12), (2.13), and (2.16) satisfies

$$M \in AC[0,\infty) \cap L^1(0,\infty), \quad M(0) = 1 - \frac{\hat{a}(0)}{a(0)^2} < 1 \quad . \tag{2.17}$$

<u>Proof</u>: That $M \in AC[0,\infty)$ is immediate. The claim concerning $M(0)$ follows from (2.14) and the facts that $M(0) = \hat{R}(0)$, $\widehat{a'}(0) = -a(0)$, and $\hat{a}(0) > 0$. To show that $M \in L^1(0,\infty)$, we proceed as in the proof of Lemma 2.3. Formally, we have

$$\hat{M}(z) = \frac{\hat{a}(z)a(0)^{-1} + \widehat{a'}(z)}{-z\widehat{a'}(z)} + \frac{1}{z} - \frac{\hat{a}(0)}{za(0)^2} \quad , \quad z \in \Pi\setminus\{0\} \quad , \tag{2.18}$$

and it is clear that $\hat{M}$ is locally analytic on $\Pi\setminus\{0\}$. To study the behavior of $\hat{M}$ near zero, we rewrite (2.18) as

$$\hat{M}(z) = \frac{\hat{a}(0)-\hat{a}(z)}{za(0)\widehat{a'}(z)} - \frac{\hat{a}(0)\hat{a}(z)}{a(0)^2\widehat{a'}(z)} \quad , \quad z \in \Pi\setminus\{0\} \quad . \tag{2.19}$$

Using Lemma 4.3 of [7] and the second part of (1.6), we find that $\hat{a}(0) - \hat{a}(z)$ has a locally analytic zero of order at least one at $z = 0$. Therefore, $\hat{M}(0)$ can be defined in such a way that $\hat{M}$ is locally analytic on Π. Finally, for z near infinity we have

$$\hat{M}(z) = \frac{\hat{a}(z)a(0)^{-1}+\widehat{a'}(z)}{1 - \widehat{a''}(z)} + \frac{1}{z} - \frac{\hat{a}(0)}{za(0)^2} \tag{2.20}$$

from which we conclude that $\hat{M}$ is locally analytic at infinity and $\hat{M}(\infty) = 0$. Therefore, by Proposition 2.3 of [7], $\hat{M}$ is the Laplace transform of a function $M \in L^1(0,\infty)$, and the desired result follows easily.

∎

Before stating the next lemma, we introduce some notation which will also be used in the next section. For $b \in L^1_{loc}[0,\infty)$, we set

$$Q(w,t,b) := \int_0^t \int_{-\infty}^{\infty} w(x,s) \int_0^s b(s-\tau)w(x,\tau)d\tau dx ds \quad , \quad \forall\, t \in [0,T] \quad , \tag{2.21}$$

for every $T > 0$ and every $w \in C([0,T]; L^2(\mathbb{R}))$. Moreover, for $T > 0$ and $0 < h < T$, we define the forward difference operator Δ_h of stepsize h (in the time variable) by

$$\Delta_h w(x,t) := w(x,t+h) - w(x,t) \quad , \quad \forall\, x \in \mathbb{R},\ t \in [0,T-h] \tag{2.22}$$

for every $w \in C([0,T]; L^2(\mathbb{R}))$.

Lemma 2.5: Assume that (1.5) holds. Then, there exists a constant $\kappa > 0$ such that

$$\int_0^t \int_{-\infty}^{\infty} w_t(x,t)^2 dx \le \kappa \int_{-\infty}^{\infty} w_t(x,0)^2 dx + \kappa Q(w_t,t,a)$$
$$+ \kappa \liminf_{h\downarrow 0} \frac{1}{h^2} Q(\Delta_h w_t,t,a) \quad \forall\, t \in [0,T] \quad , \tag{2.23}$$

for every $T > 0$ and every $w \in C^1([0,T];\, L^2(\mathbb{R}))$.

Proof: We first note that (1.5) implies $a(0) > 0$, $a \in L^2(0,\infty)$, and that there exists a constant $C > 0$ such that

$$0 \le Q(v,t,e) \le CQ(v,t,a) \quad \forall\, t \in [0,T] \tag{2.24}$$

for every $T > 0$ and every $v \in C([0,T];\, L^2(\mathbb{R}))$, where $e(t) := e^{-t}$, $t \ge 0$. Let $T > 0$, $h \in (0,T)$, and $w \in C^1([0,T];\, L^2(\mathbb{R}))$ be given. The identity

$$a(0)\Delta_h w(x,t) = a(t)\Delta_h w(x,0) + \int_0^t a(t-\tau)\Delta_h w_t(x,\tau)d\tau$$
$$- \int_0^t a'(t-\tau)\Delta_h w(x,\tau)d\tau \tag{2.25}$$

can easily be checked via integration by parts. Taking square $L^2(\mathbb{R})$ norms in (2.25) and integrating the result from 0 to t, we see that

$$a(0) \int_0^t \int_{-\infty}^{\infty} [\Delta_h w(x,s)]^2 dxds \le 3\{\int_0^{\infty} a(t)^2 dt\}\cdot\{\int_{-\infty}^{\infty} \Delta_h w(x,0)^2 dx\}$$
$$+ 3 \int_0^t \int_{-\infty}^{\infty} \{\int_0^s a(s-\tau)\Delta_h w_t(x,\tau)d\tau\}^2 dxds \tag{2.26}$$
$$+ 3 \int_0^t \int_{-\infty}^{\infty} \{\int_0^s a'(s-\tau)\Delta_h w(x,\tau)d\tau\}^2 dxds \quad .$$

It follows from (2.24), (2.26), and Lemma 4.2 of [16] that

$$a(0) \int_0^t \int_{-\infty}^{\infty} [\Delta_h w(x,s)]^2 dxds \le 3\{\int_0^{\infty} a(t)^2 dt\}\cdot\{\int_{-\infty}^{\infty} \Delta_h w(x,0)^2 dx\}$$
$$+ \bar{C}Q(\Delta_h w_t,t,a) + \bar{C}Q(\Delta_h,w,t,a) \tag{2.27}$$

where $\bar{C}$ is a constant which depends only on properties of a. To obtain the desired conclusion, we divide both sides of (2.27) by h^2 and let $h\downarrow 0$.

■

We close this section with a few remarks concerning the class of kernels which satisfy

(1.5), (1.6). As noted in Section 1, twice continuously differentiable a which satisfy $(-1)^j a^{(j)}(t) \geq 0$ for all $t > 0$, $j = 0,1,2$; $a' \not\equiv 0$ are strongly positive definite. (Corollary 2.2 of [13].) The interpretation of the integrability conditions in (1.5) and (1.6) is clear. It is not difficult to impose assumptions directly on a which will guarantee that $\mathrm{Re}\,\widehat{a'}$ does not vanish on Π.

Kernels of the form

$$a(t) := \sum_{j=1}^{N} \alpha_j e^{-\mu_j t} \quad , \quad t > 0 \quad , \tag{2.28}$$

with $\alpha_j, \mu_j > 0$ for $j = 1,\ldots,N$, which are commonly employed in applications of viscoelasticity theory, satisfy (1.5), (1.6). In fact, it is not hard to show that if a satisfies

$$\begin{aligned} a \in C^3[0,\infty),\ (-1)^j a^{(j)}(t) \geq 0 \quad \forall\, t > 0,\ j = 0,1,2,3; \\ a \not\equiv 0, \quad \int_0^\infty t a(t)dt < \infty \quad , \end{aligned} \tag{2.29}$$

then (1.5) and (1.6) hold. We remark, however, that (2.29) is by no means necessary for (1.5) and (1.6) to hold. Indeed, one readily verifies that kernels of the form $a(t) := e^{-\mu t}\cos \beta t$, $t > 0$, with μ positive satisfy (1.5), (1.6).

3. Proof of Theorem 1.1.

Let us define $\chi \in C^3(\mathbb{R})$ by

$$\chi(\xi) := \phi(\xi) - a(0)\psi(\xi) \quad \forall\, \xi \in \mathbb{R} \quad . \tag{3.1}$$

We choose a sufficiently small positive number δ and modify ϕ and ψ (and also χ accordingly by (3.1)) smoothly outside the interval $[-\delta,\delta]$ in such a way that ϕ'', ψ'' (and hence also χ'') vanish outside $[-2\delta,2\delta]$ and

$$\phi'(\xi) \geq \underline{\phi},\ \psi'(\xi) \geq \underline{\psi},\ \chi'(\xi) \geq \underline{\chi} \quad \forall\, \xi \in \mathbb{R} \quad , \tag{3.2}$$

where $\underline{\phi}$, $\underline{\psi}$, and $\underline{\chi}$ are positive constants. (This can always be accomplished in view of (1.3).) There is no harm in making this modification because we will show a posteriori that $|u_x(x,t)| < \delta$ for all $x \in \mathbb{R}$, $t > 0$.

By Lemma 2.1, (1.1), (1.2) has a unique local solution u which satisfies (2.2) on a maximal time interval $[0,T_0)$. We want to show that if (1.12) holds with μ sufficiently small, then (2.3) must also hold, and hence $T_0 = \infty$.

Estimates for the $L^2(\mathbb{R})$ norms of certain derivaties of u can be derived via energy identities. Due to the nonlinear nature of (1.1), these identities generally contain remainder terms involving integrals over time and space of quantities which are of higher algebraic order in derivatives of u. To draw useful conclusions from such an energy identity we must also obtain estimates for the remainder terms. The estimation of a remainder term often introduces new remainder terms. Therefore, the trick is to develop a closed (or self sufficient) chain of estimates. This is an especially delicate matter here due to the failure of Poincaré inequalities on unbounded intervals. The quantity E defined below has been carefully constructed for this purpose.

For $t \in [0,T_0)$, we set

$$E(t) := \max_{s\in[0,t]} \int_{-\infty}^{\infty} \{u_x^2 + u_t^2 + u_{xx}^2 + u_{xt}^2 + u_{tt}^2 + u_{xxx}^2 + u_{xxt}^2 + u_{xtt}^2 + u_{ttt}^2\}(x,s)dx$$
$$+ \int_0^t \int_{-\infty}^{\infty} \{u_{xx}^2 + u_{xt}^2 + u_{tt}^2 + u_{xxx}^2 + u_{xxt}^2 + u_{xtt}^2 + u_{ttt}^2\}(x,s)dxds \quad . \tag{3.3}$$

Our objective is to show that if (1.12) holds with μ sufficiently small then E remains bounded on $[0,T_0)$. For this purpose, it is convenient to define

$$\nu(t) := \sup_{\substack{x\in\mathbb{R}\\ s\in[0,t]}} \{u_x^2 + u_{xx}^2 + u_{xt}^2\}^{1/2}(x,s) \qquad \forall\, t \in [0,T_0) \quad . \tag{3.4}$$

To simplify the notation, we write U_0 and F in place of $U_0(u_0,u_1)$ and $F(f)$, and we use Γ to denote a (possibly large) generic positive constant which can be chosen independently of u_0, u_1, f, and T_0.

As noted in Section 1, the procedure of [2] can be used to obtain estimates for certain higher order derivatives. In fact, an inequality of the form (3.27) below can essentially be inferred from a careful examination of [2] and a few simple computations.

For the sake of completeness (and because a few changes are needed), we repeat the procedure of Dafermos and Nohel to derive (3.27).

An integration by parts in (1.1) produces

$$u_{tt}(x,t) = \chi(u_x(x,t))_x + \int_0^t a(t-\tau)\psi(u_x)_{xt}(x,\tau)d\tau$$
$$+ a(t)\psi(u_{0x}(x))_x + f(x,t) \quad . \tag{3.5}$$

We multiply (3.5) by $\psi(u_x)_{xt}$ and integrate over $\mathbb{R} \times [0,t]$, $t \in [0,T_0)$. After several integrations by parts, this yields

$$\frac{1}{2}\int_{-\infty}^{\infty}\{\psi'(u_x)u_{xt}^2 + \chi'(u_x)\psi'(u_x)u_{xx}^2\}(x,t)dx + Q(\psi(u_x)_{xt},t,a)$$
$$= \frac{1}{2}\int_{-\infty}^{\infty}\{\psi'(u_{0x})u_{1x}^2 + \chi'(u_{0x})\psi'(u_{0x})u_{0xx}^2\}(x)dx$$
$$+ \frac{1}{2}\int_0^t\int_{-\infty}^{\infty}\{\psi''(u_x)u_{xt}^3 - \chi'(u_x)\psi''(u_x)u_{xt}u_{xx}^2$$
$$+ \chi''(u_x)\psi'(u_x)u_{xt}u_{xx}^2\}(x,s)dxds \tag{3.6}$$
$$+ \int_0^t\int_{-\infty}^{\infty} a(s)\psi(u_{0x}(x))_x\psi(u_x)_{xt}(x,s)dxds$$
$$- \int_0^t\int_{-\infty}^{\infty} f_x\psi(u_x)_t(x,s)dxds \qquad \forall\ t \in [0,T_0) \quad ,$$

where Q is defined by (2.21).

To obtain the next identity, we apply the forward difference operator Δ_h to (3.5) and multiply by $\Delta_h\psi(u_x)_{xt}$. We then integrate the resulting expression over $\mathbb{R} \times [0,t]$, $t \in [0,T_0)$. After various integrations by parts, we divide by h^2 and let $h \downarrow 0$. The result of this computation is

$$
\begin{aligned}
&\frac{1}{2}\int_{-\infty}^{\infty}\{\psi'(u_x)u_{xtt}^2 + \chi'(u_x)\psi'(u_x)u_{xxt}^2\}(x,t)dx + \lim_{h\downarrow 0}\frac{1}{h^2}Q(\Delta_h\psi(u_x)_{xt},t,a) \\
&\quad = \frac{1}{2}\int_{-\infty}^{\infty}\{\psi'(u_x)u_{xtt}^2 + \chi'(u_x)\psi'(u_x)u_{xxt}^2\}(x,0)dx \\
&\quad + \int_0^t\int_{-\infty}^{\infty}\{\frac{1}{2}\psi''(u_x)u_{xt}u_{xtt}^2 - 2\psi''(u_x)u_{xt}u_{xtt}u_{ttt} \\
&\quad + \psi'''(u_x)u_{xx}u_{xt}^2u_{ttt} + \chi''(u_x)\psi'(u_x)u_{xt}u_{xxx}u_{ttt} \\
&\quad + \chi''(u_x)\psi'(u_x)u_{xx}u_{xxt}u_{xtt} + \chi'''(u_x)\psi'(u_x)u_{xx}^2u_{xt}u_{xtt} \\
&\quad - 2\chi''(u_x)\psi''(u_x)u_{xx}u_{xt}^2u_{xxt} - \chi''(u_x)\psi'''(u_x)u_{xx}^2u_{xt}^3 \\
&\quad - \frac{3}{2}\chi'(u_x)\psi''(u_x)u_{xt}u_{xxt}^2 - \chi'(u_x)\psi''(u_x)u_{xx}u_{xxt}u_{xtt} \qquad (3.7)\\
&\quad - \chi'(u_x)\psi'''(u_x)u_{xx}u_{xt}^2u_{xxt}\}(x,s)dxds \\
&\qquad - \int_0^t\int_{-\infty}^{\infty} a'(s)\psi(u_{0x}(x))_{xx}\psi(u_x)_{tt}(x,s)dxds \\
&\qquad + a(t)\int_{-\infty}^{\infty}[(\psi'(u_{0x})u_{1x})_x(x)]\psi(u_x)_{xt}(x,t)dx \\
&\qquad - a(0)\int_{-\infty}^{\infty}[(\psi'(u_{0x})u_{1x})_x(x)]^2dx \\
&\qquad - \int_0^t\int_{-\infty}^{\infty} a'(s)[(\psi'(u_{0x})(u_{1x})_x(x)]\psi(u_x)_{xt}(x,s)dxds \\
&\qquad - \int_0^t\int_{-\infty}^{\infty} f_{xt}\psi(u_x)_{tt}(x,s)dxds \qquad \forall\ t \in [0,T_0) .
\end{aligned}
$$

It is not a priori evident that $\lim_{h\downarrow 0}\frac{1}{h^2}Q(\Delta_h\psi(u_x)_{xt},t,a)$ exists for $t \in [0,T_0)$. However, the limit of each of the other terms involved in the derivation of (3.7) exists. Consequently, the limit in question exists (and is, in fact, nonnegative.) We add (3.6) to (3.7), and use Lemma 2.5 (with $w = \psi(u_x)_x$) and (3.2) to obtain a lower bound for the left hand side. After some routine estimations on the right hand side, this yields

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\{u_{xx}^2 + u_{xt}^2 + u_{xxt}^2 + u_{xtt}^2\}(x,t)dx + \int_0^t\int_{-\infty}^{\infty} u_{xxt}^2(x,s)dxds \\
&\quad \le \Gamma\{U_0 + F\} + \Gamma\{\nu(t) + \nu(t)^3\}E(t) + \Gamma\{\sqrt{U_0} + \sqrt{F}\}\sqrt{E(t)} \qquad (3.8)\\
&\qquad \forall\ t \in [0,T_0) .
\end{aligned}
$$

To give an indication of the steps involved in deriving (3.8) from (3.6) and (3.7), we show the detailed estimation of several typical terms. The reader is cautioned that there are many possible ways to carry out these and the numerous estimations which follow. We note that derivatives of ϕ, ψ, and χ of orders one through three are bounded on $\mathbb{R}$ by virtue of our modification of these functions outside $[-\delta,\delta]$.

Many of the terms from (3.6) and (3.7) can be handled in a very simple manner, e.g.

$$\begin{aligned}
&\left|\int_0^t \int_{-\infty}^{\infty} \psi''(u_x)u_{xt}u_{xtt}u_{ttt}(x,s)\,dxds\right| \\
&\qquad \le \sup_{\substack{x\in\mathbb{R}\\ s\in[0,t]}} |\psi''(u_x)u_{xt}(x,s)|\cdot\int_0^t \int_{-\infty}^{\infty} |u_{xtt}u_{ttt}(x,s)|\,dxds \\
&\qquad \le \Gamma\nu(t) \int_0^t \int_{-\infty}^{\infty} |u_{xtt}u_{ttt}(x,s)|\,dxds \qquad (3.9)\\
&\qquad \le \Gamma\nu(t) \int_0^t \int_{-\infty}^{\infty} (u_{xtt}^2 + u_{ttt}^2)(x,s)\,dxds \\
&\qquad \le \Gamma\nu(t)E(t) \qquad \forall\, t \in [0,T_0) \ .
\end{aligned}$$

A similar computation yields

$$\begin{aligned}
&\left|\int_0^t \int_{-\infty}^{\infty} f_{xt}\psi(u_x)_{tt}(x,s)\,dxds\right| \\
&\quad \le \int_0^t \int_{-\infty}^{\infty} |f_{xt}\psi'(u_x)u_{xtt}(x,s)|\,dxds + \int_0^t \int_{-\infty}^{\infty} |f_{xt}\psi''(u_x)u_{xt}^2(x,s)|\,dxds \\
&\quad \le \Gamma\cdot\left(\int_0^t \int_{-\infty}^{\infty} f_{xt}^2(x,s)\,dxds\right)^{1/2}\cdot\left(\int_0^t \int_{-\infty}^{\infty} u_{xtt}^2(x,s)\,dxds\right)^{1/2} \qquad (3.10)\\
&\qquad + \Gamma \int_0^t \int_{-\infty}^{\infty} f_{xt}^2(x,s)\,dxds + \Gamma \int_0^t \int_{-\infty}^{\infty} u_{xt}^4(x,s)\,dxds \\
&\quad \le \Gamma\sqrt{F}\cdot\sqrt{E(t)} + \Gamma F + \Gamma\nu(t)^2E(t) \qquad \forall\, t \in [0,T_0) \ .
\end{aligned}$$

To estimate terms such as the first integral on the right hand side of (3.7), we observe that the initial values of derivatives of u can be expressed in terms of u_0 and u_1 by using (1.1) if necessary. For example,

$$u_{xtt}(x,0) = \phi'(u_x)u_{xxx}(x,0) + \phi''(u_x)u_{xx}^2(x,0) + f_x(x,0) \ . \qquad (3.11)$$

Therefore,

$$\int_{-\infty}^{\infty} u_{xtt}^2(x,0)dx \le \Gamma \int_{-\infty}^{\infty} u_0'''(x)^2 dx + \Gamma \int_{-\infty}^{\infty} u_{xx}^4(x,0)dx + \Gamma \int_{-\infty}^{\infty} f_x^2(x,0)dx$$
$$\le \Gamma U_0 + \Gamma \nu(t)^2 E(t) + \Gamma F \qquad \forall\, t \in [0,T_0) \quad . \tag{3.12}$$

Of course, we could use ΓU_0^2 in place of $\Gamma\nu(t)^2 E(t)$ in (3.12). However, we already have a $\Gamma\nu(t)^2 E(t)$ term in (3.10), so there is no harm in including it here. Moreover, it simplifies matters slightly to avoid terms involving U_0^2.

The other calculations used to derive (3.8) (and our subsequent estimates) are in the same spirit as those shown above. It is useful to note that (1.5) implies $a, a' \in L^{\infty}(0,\infty)$, and that (1.8), (1.9) imply $f \in L^2([0,\infty); L^2(0,\infty))$. Moreover, we have $\int_0^{\infty} \int_{-\infty}^{\infty} f(x,t)^2 dxdt \le \frac{1}{2} F$, and clearly $\nu(t)^2 \le \nu(t) + \nu(t)^3$ for all $t \in [0,T_0)$.

Taking $L^2(\mathbb{R})$ norms in (1.1) and squaring the result, we see that

$$\int_{-\infty}^{\infty} u_{tt}^2(x,t)dx \le 3 \int_{-\infty}^{\infty} \{\phi'(u_x)^2 u_{xx}^2 + f^2\}(x,t)dx$$
$$+ 3 \int_{-\infty}^{\infty} \left(\int_0^t a'(t-\tau)\psi(u_x(x,\tau))_x d\tau\right)^2 dx \tag{3.13}$$

from which it follows easily that

$$\int_{-\infty}^{\infty} u_{tt}^2(x,t)dx \le \Gamma F + \Gamma \max_{s\in[0,t]} \int_{-\infty}^{\infty} u_{xx}^2(x,s)dx \qquad \forall\, t \in [0,T_0) \quad . \tag{3.14}$$

A similar argument gives bounds on u_{ttt}. Differentiation of (1.1) with respect to t yields

$$u_{ttt}(x,t) = \phi'(u_x)u_{xxt}(x,t) + \phi''(u_x)u_{xt}u_{xx}(x,t) + a'(t)\psi(u_{0x}(x))_x + f_t(x,t)$$
$$+ \int_0^t a'(t-\tau)[\psi'(u_x)u_{xxt} + \psi''(u_x)u_{xt}u_{xx}](x,\tau)d\tau \quad . \tag{3.15}$$

Squaring (3.15) and integrating over $\mathbb{R}$ and $\mathbb{R} \times [0,t]$, we obtain

$$\int_{-\infty}^{\infty} u_{ttt}^2(x,t)dx \le \Gamma\{U_0 + F\} + \Gamma\nu(t)^2 E(t) + \Gamma \max_{s\in[0,t]} \int_{-\infty}^{\infty} u_{xxt}^2(x,s)dx$$
$$\forall\, t \in [0,T_0) \quad , \tag{3.16}$$

and

$$\int_0^t \int_{-\infty}^{\infty} u_{ttt}^2(x,s)dxds \le \Gamma\{U_0 + F\} + \Gamma\nu(t)^2 E(t)$$
$$+ \Gamma \int_0^t \int_{-\infty}^{\infty} u_{xxt}^2(x,s)dxds \qquad \forall\, t \in [0,T_0) \quad . \tag{3.17}$$

Combining (3.8), (3.14), (3.16), and (3.17), we now have the estimate

$$\int_{-\infty}^{\infty}\{u_{xx}^2 + u_{xt}^2 + u_{tt}^2 + u_{xxt}^2 + u_{xtt}^2 + u_{ttt}^2\}(x,t)dx$$

$$+ \int_0^t \int_{-\infty}^{\infty}\{u_{xxt}^2 + u_{ttt}^2\}(x,s)dxds \tag{3.18}$$

$$\le \Gamma\{U_0 + F\} + \Gamma\{\nu(t) + \nu(t)^3\}E(t) + \Gamma\{\sqrt{U_0} + \sqrt{F}\}\sqrt{E(t)}$$

$$\forall\ t \in [0,T_0) \quad .$$

We can obtain a bound for $\int_0^t \int_{-\infty}^{\infty} u_{xtt}^2$ by interpolation. The identity

$$\int_0^t \int_{-\infty}^{\infty} v_{xtt}^2(x,s)dxds = \int_0^t \int_{-\infty}^{\infty} v_{xxt}v_{ttt}(x,s)dxds$$

$$+ \int_{-\infty}^{\infty} v_{tt}v_{xxt}(x,0)dx - \int_{-\infty}^{\infty} v_{tt}v_{xxt}(x,t)dx \qquad \forall\ t \in [0,T_0) \quad , \tag{3.19}$$

holds for all functions v having the regularity (2.2). (It is easy to give a formal derivation of (3.19) via integration by parts. It can be established rigorously using difference operators or a simple density argument.) Employing (3.19) with $v = u$, we see that

$$\int_0^t \int_{-\infty}^{\infty} u_{xtt}^2(x,s)dxds \le \Gamma U_0 + \Gamma \int_{-\infty}^{\infty}\{u_{tt}^2 + u_{xxt}^2\}(x,t)dx$$

$$+ \Gamma \int_0^t \int_{-\infty}^{\infty}\{u_{xxt}^2 + u_{ttt}^2\}(x,s)dxds \quad \forall\ t \in [0,T_0) \quad . \tag{3.20}$$

To obtain an estimate for u_{xxx}, we set

$$G(x,t) := u_{tt}(x,t) - f(x,t) + [\phi'(0) - \phi'(u_x)]u_{xx}(x,t)$$

$$+ \int_0^t a'(t-\tau)[\psi'(0) - \psi'(u_x)]u_{xx}(x,\tau)d\tau \tag{3.21}$$

and observe that (1.1) can be rewritten as

$$\phi'(0)u_{xx}(x,t) + \int_0^t a'(t-\tau)\psi'(0)u_{xx}(x,\tau)d\tau = G(x,t) \quad . \tag{3.22}$$

Using the resolvent kernel k defined by (2.10), we solve (3.22) for u_{xx} to get

$$\phi'(0)u_{xx}(x,t) = G(x,t) + \int_0^t k(t-\tau)G(x,\tau)d\tau \quad . \tag{3.23}$$

Differentiation of (3.23) with respect to x yields

$$\phi'(0)u_{xxx}(x,t) = G_x(x,t) + \int_0^t k(t-\tau)G_x(x,\tau)d\tau \quad . \tag{3.24}$$

Since $k \in L^1(0,\infty)$ (by Lemma 2.2), it follows from (3.21), (3.24), and a routine computation, that

$$\int_{-\infty}^{\infty} u_{xxx}^2(x,t)dx \le \Gamma F + \Gamma\nu(t)^2 E(t) + \max_{s\in[0,t]} \int_{-\infty}^{\infty} u_{xtt}^2(x,s)dx \qquad \forall\, t \in [0,T_0) \,, \tag{3.25}$$

and

$$\int_0^t \int_{-\infty}^{\infty} u_{xxx}^2(x,s)dxds \le \Gamma F + \Gamma\nu(t)^2 E(t) + \int_0^t \int_{-\infty}^{\infty} u_{xtt}^2(x,s)dxds \qquad \forall\, t \in [0,T_0) \,. \tag{3.26}$$

Combining (3.18), (3.20), (3.25), and (3.26), we see that

$$\begin{aligned}
&\int_{-\infty}^{\infty}\{u_{xx}^2 + u_{xt}^2 + u_{tt}^2 + u_{xxx}^2 + u_{xxt}^2 + u_{xtt}^2 + u_{ttt}^2\}(x,t)dx \\
&+ \int_0^t \int_{-\infty}^{\infty}\{u_{xxx}^2 + u_{xxt}^2 + u_{xtt}^2 + u_{ttt}^2\}(x,s)dxds \\
&\le \Gamma\{U_0 + F\} + \Gamma\{\nu(t) + \nu(t)^3\}E(t) \\
&+ \Gamma\{\sqrt{U_0} + \sqrt{F}\}\sqrt{E(t)} \qquad \forall\, t \in [0,T_0) \,.
\end{aligned} \tag{3.27}$$

It remains to obtain a similar estimate for

$$\int_{-\infty}^{\infty}\{u_x^2 + u_t^2\}(x,t)dx + \int_0^t \int_{-\infty}^{\infty}\{u_{xx}^2 + u_{xt}^2 + u_{tt}^2\}(x,s)dxds \,. \tag{3.28}$$

In particular, the remainder terms must be estimable in terms of U_0, F, $\nu(t)$ and $E(t)$. The time integral in (3.28) causes the most difficulty. As can be seen by examining the derivation of (3.27), an estimate for this term is essential.

To proceed further, we transform (1.1) to a more convenient form involving the resolvent kernel r defined by (2.12). This transformation was motivated by an idea of MacCamy [9], [10]. As explained in Section 2, we assume without loss of generality that $a'(0) = -1$.

Differentiating (1.1) with respect to t, we get

$$u_{ttt}(x,t) = \phi(u_x(x,t))_{xt} - \psi(u_x(x,t))_x + \int_0^t a''(t-\tau)\psi(u_x(x,\tau))_x d\tau + f_t(x,t) \tag{3.29}$$

Solving (3.29) for $\psi(u_x)_x$ and rearranging the terms, we obtain

$$\begin{aligned}
u_{ttt}(x,t) &= \phi(u_x(x,t))_{xt} - \psi(u_x(x,t))_x + f_t(x,t) \\
&+ \int_0^t r(t-\tau)[\phi(u_x)_{xt} - u_{ttt} + f_t](x,\tau)d\tau \,,
\end{aligned} \tag{3.30}$$

or, setting $\alpha := a(0)$ and using (2.13),

$$u_{ttt} + \alpha u_{tt} = \phi(u_x)_{xt} + \alpha\chi(u_x)_x + f_t + \alpha f + R*(\phi(u_x)_{xt} - u_{ttt} + f_t) \quad , \tag{3.31}$$

or finally,

$$\begin{aligned} u_{ttt} + \alpha u_{tt} &= \phi(u_x)_{xt} + \alpha\chi(u_x)_x + f_t + \alpha f \\ &\quad + [R*(\phi(u_x)_x - u_{tt} + f)]_t \ , \end{aligned} \tag{3.32}$$

where the $*$ denotes convolution with respect to the time variable on $[0,t]$, i.e.

$$(R*v)(x,t) := \int_0^t R(t-\tau)v(x,\tau)d\tau \quad . \tag{3.33}$$

(In the above calculations, we have made use of the fact that $[\phi(u_x)_x - u_{tt} + f](x,0) \equiv 0$ which follows from (1.1).) Recall that $\alpha := a(0) > 0$ and that $R \in L^1(0,\infty)$ by Lemma 2.3.

Let us set

$$W(\xi) := \int_0^\xi \chi(x)ds \qquad \forall\ \xi \in \mathbb{R} \ , \tag{3.34}$$

and note that

$$W(\xi) \geq \frac{1}{2}\underline{\chi}\xi^2 \qquad \forall\ \xi \in \mathbb{R} \tag{3.35}$$

by virtue of (1.3), (3.2), and (3.34). We multiply (3.32) by u_t and integrate over $\mathbb{R} \times [0,t]$, as before, to get

$$\begin{aligned} &\alpha \int_{-\infty}^{\infty}\{\frac{1}{2}u_t^2 + W(u_x)\}(x,t)dx + \int_0^t \int_{-\infty}^{\infty} \phi'(u_x)u_{xt}^2(x,s)dxds \\ &\quad - \int_0^t \int_{-\infty}^{\infty} u_{tt}^2(x,s)dxds + \int_0^t \int_{-\infty}^{\infty} u_{tt}[R*(\phi(u_x)_x - u_{tt} + f)](x,s)dxds \\ &\quad = \int_{-\infty}^{\infty}\{\frac{1}{2}\alpha u_t^2 + \alpha W(u_x) + u_t u_{tt} - fu_t\}(x,0)dx \\ &\qquad + \int_{-\infty}^{\infty}\{fu_t - u_t u_{tt}\}(x,t)dx \\ &\qquad + \int_{-\infty}^{\infty} u_t[R*(\phi(u_x)_x - u_{tt} + f)](x,t)dx \\ &\qquad + \int_0^t \int_{-\infty}^{\infty}\{\alpha fu_t - fu_{tt}\}(x,s)dxds \qquad \forall\ t \in [0,T_0) \ . \end{aligned} \tag{3.36}$$

Next, we multiply (3.5) by u_{tt} and integrate over $\mathbb{R} \times [0,t]$. This yields

$$\int_0^t \int_{-\infty}^{\infty} u_{tt}^2(x,s)dxds - \int_0^t \int_{-\infty}^{\infty} \chi'(u_x)u_{xt}^2(x,s)dxds$$

$$= \int_{-\infty}^{\infty} \chi'(u_x)u_x u_{xt}(x,0)dx - \int_{-\infty}^{\infty} \chi'(u_x)u_x u_{xt}(x,t)dx$$

$$+ \int_0^t \int_{-\infty}^{\infty} \{fu_{tt} + \chi''(u_x)u_x u_{xt}^2\}(x,s)dxds \tag{3.37}$$

$$+ \int_0^t \int_{-\infty}^{\infty} [a(s)\psi(u_{0x}(x))_x]u_{tt}(x,s)dxds$$

$$+ \int_0^t \int_{-\infty}^{\infty} u_{tt}[a*\psi(u_x)_{xt}](x,s)dxds \qquad \forall\ t \in [0,T_0) \quad .$$

Adding (3.36) to (3.37), we find that

$$\alpha \int_{-\infty}^{\infty}\{\tfrac{1}{2} u_t^2 + W(u_x)\}(x,t)dx + a(0) \int_0^t \int_{-\infty}^{\infty} \psi'(u_x)u_{xt}^2(x,s)dxds$$

$$+ \int_0^t \int_{-\infty}^{\infty} u_{tt}[R*(\phi(u_x)_x - u_{tt} + f)](x,s)dxds$$

$$= \int_{-\infty}^{\infty}\{\tfrac{1}{2} \alpha u_t^2 + \alpha W(u_x) + u_t u_{tt} + \chi'(u_x)u_x u_{xt} - fu_t\}(x,0)dx$$

$$+ \int_{-\infty}^{\infty}\{fu_t - u_t u_{tt} - \chi'(u_x)u_x u_{xt}\}(x,t)dx$$

$$+ \int_{-\infty}^{\infty} u_t[R*(\phi(u_x)_x - u_{tt} + f)](x,t)dx \tag{3.38}$$

$$+ \int_0^t \int_{-\infty}^{\infty}\{\alpha fu_t + \chi''(u_x)u_x u_{xt}^2\}(x,s)dxds$$

$$+ \int_0^t \int_{-\infty}^{\infty} [a(s)\psi(u_{0x}(x))_x]u_{tt}(x,s)dxds$$

$$+ \int_0^t \int_{-\infty}^{\infty} u_{tt}[a*\psi(u_x)_{xt}](x,s)dxds \qquad \forall\ t \in [0,T_0) \quad .$$

The crucial term to analyze is

$$\Phi(t) := \int_0^t \int_{-\infty}^{\infty} u_{tt}[R*(\phi(u_x)_x - u_{tt} + f)](x,s)dxds \quad . \tag{3.39}$$

(The other terms in (3.38) are favorable or can be estimated routinely.) We see from (3.1) and (3.5) that

$$\phi(u_x)_x - u_{tt} + f = a(0)\psi(u_x)_x - a(t)\psi(u_{0x})_x - a*\psi(u_x)_{xt} \quad , \tag{3.40}$$

and substitution into (3.39) yields

$$\Phi(t) = a(0) \int_0^t \int_{-\infty}^{\infty} u_{tt}[R*\psi(u_x)_x](x,s)\,dxds$$

$$- \int_0^t \int_{-\infty}^{\infty} [R*a](s)\psi(u_{0x}(x))_x u_{tt}(x,s)\,dxds \tag{3.41}$$

$$- \int_0^t \int_{-\infty}^{\infty} u_{tt}[R*a*\psi(u_x)_{xt}](x,s)\,dxds \quad .$$

Since R, $a \in L^1(0,\infty)$ and we already have estimates on third order derivatives of u, the last two terms in (3.41) cause no difficulties. However, the first term on the right hand side of (3.41) requires special attention.

Employing the kernel M defined by (2.16), we find that

$$R*\psi(u_x)_x = -M(0)\psi(u_x)_x + M\psi(u_{0x})_x + M*\psi(u_x)_{xt} \quad . \tag{3.42}$$

Recall that $M \in L^1(0,\infty)$ and $M(0) < 1$. We observe further that

$$\int_0^t \int_{-\infty}^{\infty} \psi(u_x)_x u_{tt}(x,s)\,dxds = \int_0^t \int_{-\infty}^{\infty} \psi'(u_x)u_{xt}^2(x,s)\,dxds$$

$$+ \int_{-\infty}^{\infty} \psi(u_x)u_{xt}(x,0)\,dx - \int_{-\infty}^{\infty} \psi(u_x)u_{xt}(x,t)\,dx \tag{3.43}$$

$$\forall\, t \in [0,T_0) \quad ,$$

using integration by parts. Combining (3.41), (3.42), and (3.43), we arrive at the following expression for Φ:

$$\begin{aligned}\Phi(t) = {} & -a(0)M(0) \int_0^t \int_{-\infty}^{\infty} \psi'(u_x)u_{xt}^2(x,s)\,dxds \\ & + a(0) \int_{-\infty}^{\infty} \psi(u_x)u_{xt}(x,0)\,dx - a(0) \int_{-\infty}^{\infty} \psi(u_x)u_{xt}(x,t)\,dx \\ & + \int_0^t \int_{-\infty}^{\infty} [M - (R*a)](s)\psi(u_{0x}(x))_x u_{tt}(x,s)\,dxds \\ & + \int_0^t \int_{-\infty}^{\infty} u_{tt}\{[M - (R*a)]*\psi(u_x)_{xt}\}(x,s)\,dxds \quad .\end{aligned} \tag{3.44}$$

Since $M(0) < 1$, the first integral in the above expression can be absorbed by the second integral on the left hand side of (3.38). Moreover, the remaining terms can be handled rather easily. (Note that $[M - (R*a)] \in L^1(0,\infty)$, since $M,R,a \in L^1(0,\infty)$.) After substitution of (3.44) into (3.38), and a long computation, we obtain

$$\int_{-\infty}^{\infty}(u_t^2 + u_x^2)(x,t)dx + \int_0^t \int_{-\infty}^{\infty} u_{xt}^2(x,s)dxds$$

$$\leq \Gamma\{U_0 + F\} + \Gamma\nu(t)E(t)$$

$$+ \Gamma(\int_0^t \int_{-\infty}^{\infty} u_{tt}^2(x,s)dxds)^{1/2} \cdot (\int_0^t \int_{-\infty}^{\infty} u_{xxt}^2(x,s)dxds) \tag{3.45}$$

$$+ \Gamma \max_{s\in[0,t]} \int_{-\infty}^{\infty}(u_{xx}^2 + u_{xt}^2 + u_{tt}^2)(x,s)dx \qquad \forall\, t \in [0,T_0) \quad .$$

In the derivation of (3.45), we have used the simple algebraic inequality

$$|AB| \leq \varepsilon A^2 + \frac{1}{4\varepsilon} B^2 \qquad \forall\, \varepsilon > 0 \quad , \tag{3.46}$$

to handle several terms. For example, observe that

$$|\int_{-\infty}^{\infty} \chi'(u_x)u_x u_{xt}(x,t)dt| \leq \bar{\chi} \int_{-\infty}^{\infty}|u_x u_{xt}(x,t)|dx$$

$$\leq \varepsilon\bar{\chi} \int_{-\infty}^{\infty} u_x^2(x,t)dx + \frac{\bar{\chi}}{4\varepsilon} \int_{-\infty}^{\infty} u_{xt}^2(x,t)dx \qquad \forall\, t \in [0,T_0) \tag{3.47}$$

for every $\varepsilon > 0$, where $\bar{\chi} := \sup_{\xi\in\mathbb{R}} |\chi'(\xi)|$. On account of (3.35), $\varepsilon\bar{\chi} \int_{-\infty}^{\infty} u_x^2$ can be absorbed by the first integral on the left hand side of (3.38) if ε is sufficiently small. The size of the coefficient $\frac{\bar{\chi}}{4\varepsilon}$ is unimportant because we already have an estimate for $\int_{-\infty}^{\infty} u_{xt}^2$. Moreover, we have made essential use of the assumption $f \in L^1([0,\infty); L^2(\mathbb{R}))$ to estimate $\int_0^t \int_{-\infty}^{\infty} fu_t$ since it does not seem possible to obtain a time independent bound for $\int_0^t \int_{-\infty}^{\infty} u_t^2$.

It follows from (3.37) and a simple computation that

$$\int_0^t \int_{-\infty}^{\infty} u_{tt}^2(x,s)dxds \leq \Gamma\{U_0 + F\} + \Gamma\nu(t)E(t)$$

$$+ \Gamma(\int_0^t \int_{-\infty}^{\infty} u_{tt}^2(x,s)dxds)^{1/2} \cdot (\int_0^t \int_{-\infty}^{\infty} u_{xxt}^2(x,s)dxds)^{1/2}$$

$$+ \Gamma \max_{s\in[0,t]} \int_{-\infty}^{\infty} u_{tt}^2(x,s)dx \tag{3.48}$$

$$+ \Gamma \int_0^t \int_{-\infty}^{\infty} u_{xt}^2(x,s)dxds \qquad \forall\, t \in [0,T_0) \quad .$$

Combining (3.45) and (3.48), we thus obtain

$$\int_{-\infty}^{\infty}\{u_x^2 + u_t^2\}(x,s)dx + \int_0^t \int_{-\infty}^{\infty}\{u_{xt}^2 + u_{tt}^2\}(x,s)dxds$$

$$\leq \Gamma\{U_0 + F\} + \Gamma\nu(t)E(t)$$

$$+ \Gamma(\int_0^t \int_{-\infty}^{\infty} u_{tt}^2(x,s)dxds)^{1/2} \cdot (\int_0^t \int_{-\infty}^{\infty} u_{xxt}^2(x,s)dxds)^{1/2} \tag{3.49}$$

$$+ \Gamma \max_{s\in[0,t]} \int_{-\infty}^{\infty}\{u_{xx}^2 + u_{xt}^2 + u_{tt}^2\}(x,s)dxds \quad \forall\, t \in [0,T_0) \ ,$$

and using (3.46) with ε sufficiently small, and (3.27),

$$\int_{-\infty}^{\infty}\{u_x^2 + u_t^2\}(x,t)dx + \int_0^t \int_{-\infty}^{\infty}\{u_{xx}^2 + u_{xt}^2 + u_{tt}^2\}(x,s)dxds$$

$$\leq \Gamma\{U_0 + F\} + \Gamma\{\nu(t) + \nu(t)^3\}E(t) \tag{3.50}$$

$$+ \Gamma\{\sqrt{U_0} + \sqrt{F}\}\sqrt{E(t)} \quad \forall\, t \in [0,T_0) \ .$$

To obtain our last estimate, we go back to (3.23). Using (3.23), (3.21), and the fact that $k \in L^1(0,\infty)$, we deduce that

$$\int_0^t \int_{-\infty}^{\infty} u_{xx}^2(x,s)dxds \leq \Gamma F + \Gamma\nu(t)^2 E(t) + \Gamma \int_0^t \int_{-\infty}^{\infty} u_{tt}^2(x,s)dxds \tag{3.51}$$

$$\forall\, t \in [0,T_0) \ .$$

Combining (3.50) and (3.51), and adding the result to (3.27), we conclude that

$$E(t) \leq \Gamma\{U_0 + F\} + \Gamma\{\nu(t) + \nu(t)^3\}E(t) + \Gamma\{\sqrt{U_0} + \sqrt{F}\}\sqrt{E(t)} \tag{3.52}$$

$$\forall\, t \in [0,T_0) \ ,$$

and using (3.46), we finally arrive at an estimate of the form

$$E(t) \leq \overline{\Gamma}\{U_0 + F\} + \overline{\Gamma}\{\nu(t) + \nu(t)^3\}E(t) \quad \forall\, t \in [0,T_0) \ , \tag{3.53}$$

where $\overline{\Gamma}$ denotes a fixed positive constant which can be chosen independently of u_0, u_1, f, and T_0.

We choose $\overline{E}, \overline{\mu} > 0$ such that

$$\overline{E} < \delta^2, \quad \overline{\Gamma}\{(2\overline{E})^{1/2} + (2\overline{E})^{3/2}\} < \frac{1}{2}, \quad \overline{\Gamma}\mu^2 < \frac{1}{4}\overline{E} \ . \tag{3.54}$$

(Here δ is the constant that was introduced in the first paragraph of this section.)

Suppose now that (1.12) holds with the above choice of μ. It follows from the Sobolev embedding theorem that

$$\nu(t) < \sqrt{2E(t)} \qquad \forall\ t \in [0,T_0) \quad . \tag{3.55}$$

We therefore conclude from (3.53), (3.54), and (3.55) that for any $t \in [0,T_0)$ with $E(t) < \bar{E}$, we actually have $E(t) < \frac{1}{2}\bar{E}$. Consequently, by continuity,

$$E(t) < \frac{1}{2}\bar{E} \qquad \forall\ t \in [0,T_0) \tag{3.56}$$

provided that $E(0) < \frac{1}{2}\bar{E}$.

We can always choose a smaller $\mu > 0$ (if necessary such that (1.12) implies $E(0) < \frac{1}{2}\bar{E}$. (Observe that (3.54) still holds if the size of μ is reduced.) Thus, if (1.12) is satisfied with our revised choice of μ then (3.56) holds. This implies $T_0 = \infty$ by Lemma 2.1. In addition, it immediately yields (1.13) and (1.14) from which (1.15) and (1.16) follow by standard embedding inequalities. Finally, we note that

$$|u_x(x,t)| < \nu(t) < (\bar{E})^{1/2} < \delta \qquad \forall\ x \in R,\ t > 0 \quad . \tag{3.57}$$

by (3.4), (3.55), (3.56), and (3.54). The proof of Theorem 1.1 is complete. ■

REFERENCES

1. Dafermos, C. M., and J. A. Nohel, Energy methods for nonlinear hyperbolic Volterra integrodifferential equations, Comm. PDE 4 (1979), 219-278.
2. Dafermos, C. M., and J. A. Nohel, A nonlinear hyperbolic Volterra equation in viscoelasticity, Am. J. Math. Supplement (1981), 87-116.
3. Gripenberg, G., Nonexistence of smooth solutions for shearing flows in a nonlinear viscoelastic fluid, SIAM J. Math. Anal. 13 (1982), 954-961.
4. Hattori, H., Breakdown of smooth solutions in dissipative nonlinear hyperbolic equations, Q. Appl. Math. 40 (1982), 113-127.
5. Hrusa, W. J., and J. A. Nohel, Global existence and asymptotics in one-dimensional nonlinear viscoelasticity, Proc. 5th Symp. on Trends in Applications of Pure Mathematics to Mechanics, Springer Lecture Notes in Physics (to appear).
6. Hrusa, W. J., and M. Renardy, On a class of quasilinear integrodifferential equations with singular kernels, (to appear).
7. Jordan, G. S., O. J. Staffans and R. L. Wheeler, Local analyticity in weighted L^1-spaces and applications to stability problems for Volterra equations, Trans. Amer. Math. Soc. 174 (1982), 749-782.
8. Lax, P. D., Development of singularities in solutions of nonlinear hyperbolic partial differential equations, J. Math. Phys. 5 (1964), 611-613.
9. MacCamy, R. C., An integro-differential equation with application in heat flow, Q. Appl. Math. 35 (1977), 1-19.
10. MacCamy, R. C., A model for one-dimensional nonlinear viscoelasticity, Q. Appl. Math. 35 (1977), 21-33.
11. MacCamy, R. C. and V. J. Mizel, Existence and nonexistence in the large of solutions of quasilinear wave equations, Arch. Rational Mech. Anal. 25 (1967), 299-320.
12. Malek-Madani, R. and J. A. Nohel, Formation of singularities for a conservation law with memory, SIAM J. Math. Anal., (to appear).
13. Nohel, J. A. and D. F. Shea, Frequency domain methods for Volterra equations, Advances in Math. 22 (1976), 278-304.

14. Slemrod, M., Instability of steady shearing flows in a nonlinear viscoelastic fluid, Arch. Rational Mech. Anal. 68 (1978), 211-225.

15. Staffans, O., Nonlinear Volterra integral equations with positive definite kernels, Proc. Amer. Math. Soc. 5 (1975), 103-108.

16. Staffans, O., On a nonlinear hyperbolic Volterra equations, SIAM J. Math. Anal. 11 (1980), 793-812.

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We study the initial value problem for a nonlinear hyperbolic Volterra equation which models the motion of an unbounded viscoelastic bar. Under physically motivated assumptions, we establish the existence of a unique, globally defined, classical solution provided the initial data are sufficiently smooth and small. We also discuss boundedness and asymptotic behavior. Our analysis is based on energy estimates in conjunction with properties of strongly positive definite kernels.